TOPIC 5: Indefinite Integrals and Techniques of Integration

1. INDEFINITE INTEGRAL

Definition: Antiderivatives

A function F is said to be an **antiderivative** of a function f if

$$F^{'}(x) = f(x)$$

at every number x where $f(x)$ is defined.

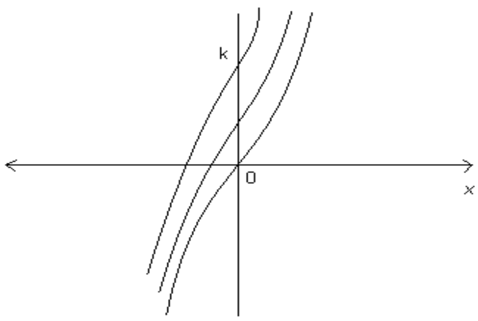
> If $F^{'}(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$.
>
> F is an antiderivative of f iff f is the derivative of F.

Example 1:

Is $f(x) = 3x^2 + 2$ the derivative of a certain function? What is an antiderivative of f ?

$F(x) = x^3 + 2x + 4$ and $G(x) = x^3 + 2x$ are both antiderivatives of $f(x)$.
Note that $F(x) = G(x) + C$, with $C = 4$.

$y = x^3 + 2x + k$



In fact every antiderivative of $f(x)$ with $f(x) = 3x^2 + 2$ *has the form* $x^3 + 2x + C$, *for some constant* C.

Theorem:

If $F(x)$ and $G(x)$ are antiderivatives of a function $f(x)$ on an interval, then there is a constant C such that $F(x) - G(x) = C$.

(Two antiderivatives of a function can differ only by a constant.)

If G is an antiderivative of f and $F(x) = G(x) + C$ for a constant C, then G is also an antiderivative of f .

A special symbol is used to denote the collection of all antiderivatives of a function f .

<u>Definition: Indefinite Integrals</u>

We call the set of all antiderivatives of a function the **indefinite integral** of the function. The indefinite integral of a function f using the variable x is denoted by

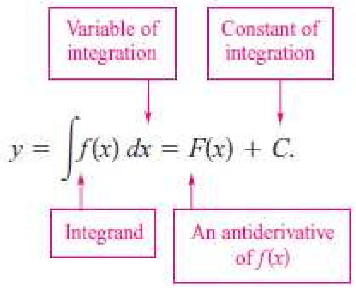
$\int f(x)dx$. [Read as "the indefinite integral of $f(x)$ with respect to x".]

The symbol $\int$ is an **integral sign.** The function f is the **integrand** of the integral, and x is the **variable of integration.**

If ***F'(x) = f(x)***, then

$$\int f(x)dx = F(x) + C$$

where C is an arbitrary constant called the **constant of integration**.



When we find $F(x)+C$, we say that we **integrate** $f(x)$ **with respect to** x and evaluate the integral $\int f(x)dx$.

<u>Example 2:</u>

(a) $\int 2x dx = x^2 + C$ (b) $\int 4x^3 dx = x^4 + C$ (c) $\int 10x^4 dx =$

(d) $\int x^4 dx =$ (e) $\int 7dx =$ (f) $\int \cos x\, dx =$

[**<u>To Students</u>**: Whenever you have carried out an integration, how can you check if you have done it correctly?]

2. INTEGRATION RULES

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$ respectively, and k is an arbitrary number. Then:

1. $\int f(x)dx = F(x) + C$ (Note that this is equivalent to $\int \left[\frac{d}{dx} F(x) \right] dx = F(x) + C$)
2. Constant Multiple Rule: $\int kf(x)dx = k\int f(x)dx = kF(x) + C$
3. Rule for Negatives: $\int -f(x)dx = -\int f(x)dx = -F(x) + C$
4. Sum and Difference Rule:
$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx = F(x) \pm G(x) + C$$

Be smart in using the above rules, as illustrated by the following example from Thomas' Calculus.

Example 3

Evaluate $\int (x^2 - 2x + 5)dx$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \underbrace{\frac{x^3}{3} - x^2 + 5x}_{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\ &= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx \\ &= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\ &= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the possible antiderivatives there are.

For this reason, we recommend that you go right to the final form even if you elect to integrate term-by-term. Write

$$\begin{aligned}\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\ &= \frac{x^3}{3} - x^2 + 5x + C.\end{aligned}$$

Find the simplest antiderivative you can for each part and add the arbitrary constant of integration at the end.

(T*)

For integration:

1. Know basic integration rules
2. Remember a few basic formulas based on your knowledge from differentiation
3. Know how to use substitution
4. Know how to use partial fractions + trigonometric identities
5. Know integration by parts

3. BASIC INTEGRATION FORMULAS

There is no real need to memorize any so-called basic integration formula if you know basic differentiation well.

<u>What you must remember from differentiation:</u>

Differentiation you should know	**<u>Integration Formulas</u>** (*with some understanding and practice, you should be able to remember without memorizing*)
$\frac{d}{dx}x^n = nx^{n-1}, n \in R$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \ne -1$)
$\frac{d}{dx}\sin x = \cos x.$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x.$	$\int \sin x dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x.$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}e^x = e^x$	$\int e^x dx = e^x + C$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$ (for $x > 0$); $\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$ (for $x \ne 0$)	$\int \frac{1}{x} dx = \ln\lvert x\rvert + C$
Constant Multiple Rule Sum Rule Product Rule Quotient Rule	
The chain rule	

<u>Example 4:</u>

(a) $\int (4x^{2.3} + x^{-1})dx = \frac{4x^{3.3}}{3.3} + \ln|x| + C$

(b) $\int (6x^{-1} + 7x^{-4})dx = 6\ln|x| + \frac{7x^{-3}}{-3} + C = 6\ln|x| - \frac{7}{3}x^{-3} + C$

(c) $\int (10e^x - 5)dx$ (d) $\int (3\sin x + 4\cos x)dx$

(e) $\int \left(\frac{7}{\sqrt{x}} - 3\sec^2 x\right)dx$ (f) $\int \sqrt{x}(2x^{-1} + 7x^4)dx$

(g) $\int (x^3 - 2)^2 dx$ (h) $\int \frac{x^2 - 1}{\sqrt{x}} dx$

(i) $\int \frac{x^2 - 1}{\sqrt{x}} dx$ (j) $\int (x^3 - 2)^2 dx$

Formulas that need no memorization if you know the chain rule for differentiation.

	First, complete this column, using the chain rule for differentiation.	You'll then be able to complete this column; so you can produce your own table. NO NEED to memorize too many things!
A	$\frac{d}{dx}\left(e^{kx}\right)=$	$\int e^{kx}\ dx=$
B	$\frac{d}{dx}(\cos kx)=$	$\int \sin kx\ dx=$
C	$\frac{d}{dx}(\sin kx)=$	$\int \cos kx\ dx=$
D	$\frac{d}{dx}(\tan kx)=$	$\int \sec^2 kx\ dx=$
		Where do all these formulas come from? Think of them as guesswork; to find an antiderivative of f, all we need to do is to come up with a function *whose derivative is f*. These formulas could also be obtained by the substitution method discussed in the next section.

<u>**Example 5:**</u>

(a) $\int\left(\sin 3t-2e^{-3t}\right)dt$ (b) $\int(6e^{2x}+\sec^2 5x\ )dx$ (c) $\int\left(5\cos 2u-2e^{4u}\right)du$

<u>**No need to memorize these formulas.**</u>
<u>**Learn to derive them yourself; you are then doing some real mathematics**</u>.

The following can be obtained via substitution rule.	
$\int(ax+b)^n\,dx$, for $n\neq -1$	$\int\frac{1}{ax+b}\,dx$

The following can be obtained via integration by parts (*to be discussed later*).
$\int \ln x\ \ dx = x\ln x - x + C$

Some authors/teachers like to list the following for student to memorize.

1. $\int[g(x)]^n g'(x)dx=\frac{1}{n+1}[g(x)]^{n+1}+C$ for $n\neq -1$

2 $\int\frac{g'(x)}{g(x)}dx=\ln g(x)+C$

3 $\int e^{g(x)}g'(x)dx=e^{g(x)}+C$

[*Can be obtained via substitution rule; you should know them through practice; no real need to memorize*.]

<u>**Example 6**</u>: (Please see the next section)

(a) $\int(ax+b)^n\,dx$ (b) $\int\frac{1}{ax+b}\,dx$

4. INTEGRATION BY SUBSTITUTION
(also called integration by change of variable)

With the right choice of substitution, an integral in one variable can be transformed into another one (involving a new variable) that is easier to evaluate.

Example 7:

$\int (2x^3+1)^4(6x^2)dx$ [We are going to use the substitution $u=2x^3+1$.]

Let $u=2x^3+1$. Then $du=\frac{du}{dx}dx=6x^2dx$.

[Note : Some people would write $du=6x^2dx$ directly. To those who have some difficulty seeing it, you are advised to write the differentiation step $\frac{du}{dx}=6x^2$ first followed by $du=6x^2dx$.]

$\int (2x^3+1)^4(6x^2)dx$

$=\int (\overbrace{2x^3+1}^{u})^4(\overbrace{6x^2\,dx}^{du})$

$=\int u^4du$ [The variable of integration is changed from x to u.]

$=\frac{u^5}{5}+C$ [This expression involves u.]

$=\frac{(2x^3+1)^5}{5}+C$. [The original integral involves x, so we transform to an expression involving x.]

[You can check the correctness of this result by finding $\frac{d}{dx}\left[\frac{(2x^3+1)^5}{5}+C\right]$.]

Example 6: (Revisited)

(a) $\int (ax+b)^n dx$ (b) $\int \frac{1}{ax+b}dx$

Example 8:

$\int \frac{2x}{\sqrt{x+3}}dx$ Let $u=\sqrt{x+3}$. Then $u^2=x+3$, so $x=u^2-3$ and $dx=2u\,du$

$$\int \frac{2x}{\sqrt{x+3}}dx=\int \frac{2(u^2-3)}{u}2u\,du=\int 4(u^2-3)du=\ldots$$

Theorem – The Subsitution Rule
If $u = g(x)$ is differentiable function whose range is an interval I and f is a continuous function on I, then

$$\int f(g(x)) \cdot g'(x)dx = \int f(u)du$$

Example 9:

Use the Method of Substitution to evaluate the indefinite integrals

(a) $\int 2x\sqrt{1+x^2}\,dx$ (b) $\int \sin^4 x \cos x\,dx$, $u = \sin x$ (c) $\int \frac{x}{x^2+1}dx, u = x^2+1$,

(d) $\int \frac{x+4}{(x^2+8x)^3}dx$ (e) $\int x^2 e^{x^3}\,dx$

Solution (a):

$$\text{Let } u = 1+x^2, \frac{du}{dx} = 2x, du = 2xdx, \quad xdx = \frac{1}{2}du$$

$$\int 2x\sqrt{1+x^2}\,dx = \int 2u^{\frac{1}{2}} \cdot \frac{1}{2}du = \int u^{\frac{1}{2}}du$$

$$= \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$

$$= \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$$

5. INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

To see how the method of partial fractions works in general, let's consider a rational function $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials.

If the degree of P is less than the degree of Q, such a rational function is called *proper* and it is possible to express f as a sum of simpler fractions.

If the degree of P is greater than or equal to the degree of Q (in this case, f is called *improper*), then f can be expressed as a sum of a polynomial and a proper rational function through long division.

$f(x) = \frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$, where $R(x)$ is the remainder of the long division and has degree less than that of Q.

For a full discussion on partial fraction, you need to refer to your earlier study.

*******For this course, we shall consider only the case where $Q(x)$ can be written as a product of distinct linear factors.******* (as illustrated by the following examples)

<u>Example 10:</u>

(a) $\int \frac{1}{x^2-9}dx$ (b) $\int \frac{x+4}{2x^3-3x^2-2x}dx$

Solution (a)

$$x^2-9=(x-3)(x+3)$$

$$\frac{1}{x^2-9}=\frac{1}{(x-3)(x+3)}=\frac{A}{x-3}+\frac{B}{x+3}$$

$$\frac{1}{(x-3)(x+3)}=\frac{A(x+3)+B(x-3)}{(x-3)(x+3)}$$

$$A(x+3)+B(x-3)=1$$

TWO ways to determine the values of A and B

Method (i) (by substituting different values for x)		Method (ii) (by equating coefficients)
$A(x+3)+B(x-3)=1$ When $x=-3$, $-6B = 1$; so $B=-\frac{1}{6}$. When $x=3$, $6A = 1$; so $A=\frac{1}{6}$. So, $\frac{1}{x^2-9}=\frac{1}{6}\cdot\frac{1}{x-3}-\frac{1}{6}\cdot\frac{1}{x+3}$		$A(x+3)+B(x-3)=1$ $Ax+3A+Bx-3B=1$ $(A+B)x+3A-3B=1$ Equating coefficients: $\begin{cases} A+B=0 \\ 3A-3B=1 \end{cases}$ (*More steps needed here*.) Solving, $A=\frac{1}{6}$, $B=-\frac{1}{6}$ So, $\frac{1}{x^2-9}=\frac{1}{6}\cdot\frac{1}{x-3}-\frac{1}{6}\cdot\frac{1}{x+3}$

$$\int\frac{1}{x^2-9}dx=\frac{1}{6}\int\frac{1}{x-3}dx-\frac{1}{6}\int\frac{1}{x+3}dx=\frac{1}{6}\ln|x-3|-\frac{1}{6}|x+3|+c$$

(b) $\int \frac{x+4}{2x^3-3x^2-2x}dx$

$$2x^3-3x^2-2x=x(2x^2-3x-2)=x(2x+1)(x-2)$$

$$\frac{x+4}{x(2x+1)(x-2)}=\frac{A}{x}+\frac{B}{2x+1}+\frac{C}{x-2}$$

$$x+4=A(2x+1)(x-2)+Bx(x-2)+x(2x+1)$$

Then, ……(*More steps needed here*.)

A = B = C =

(Continue)

6. INTEGRATION INVOLVING TRIGONOMETRIC IDENTITIES

In Topic 2 (Complex Numbers and Trigonometric Identities), Euler's formula has been used to derive trigonometric identities that

(i) express $\sin mx \cos nx$, $\cos mx \cos nx$ or $\sin mx \sin nx$ as a sum/difference of sines/cosines.

(ii) express a power of $\sin mx$ or $\cos nx$, as a sum of terms involving sines/cosines (not any higher power of it) and/or constant.

We shall confine ourselves to integrands of types described in (i) and (ii).

For other integrands involving trigonometric functions, appropriate identities will be provided. [It is not necessary for you to memorize too many identities.]

<u>Example 11</u>

(a) $\int \sin 3x \cos 4x \, dx$ (b) $\int \cos^3 5x \, dx$

Solution: (a)

In example 3, section 2.4 of Topic 2, Euler's formula was use to derive:

$$\sin 3x \cos 4x = \frac{1}{2}(\sin 7x - \sin x)$$

So, $$\int \sin 3x \cos 4x \, dx = \frac{1}{2}\int (\sin 7x - \sin x)\, dx = \frac{1}{2}(-\frac{1}{7}\cos 7x + \cos x) + c$$

$$= -\frac{1}{14}\cos 7x + \frac{1}{2}\cos x + c$$

(b) From Euler's formula, we have $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$.

$$\cos^3\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^3 = \frac{e^{i3\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-i3\theta}}{8} = \frac{e^{i3\theta} + e^{-i3\theta} + 3(e^{i\theta} + e^{-i\theta})}{8}$$

$$= \frac{2\cos 3\theta + 6\cos\theta}{8} = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

So $$\int \cos^3 5x \, dx = \int \left[\frac{1}{4}\cos 15x + \frac{3}{4}\cos 5x\right] dx = \frac{1}{60}\sin 15x + \frac{3}{20}\sin 5x + C$$

7. INTEGRATION BY PARTS

If u and v are differentiable functions, then

$$\int \frac{d}{dx}[u(x)v(x)]\,dx = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

$$\int u(x)v'(x)dx = u(x)v(x) - \int v(x)u'(x)dx \quad \text{or} \quad \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx\,.$$

This equation can be written in a simpler form:

$$\int udv = uv - \int vdu$$

An integral like $\int udv$ is transformed into a form involving $\int vdu$; let's hope that $\int vdu$ is easier to handle than $\int udv$.

<u>Examples 12:</u>
Use Integration by Parts to evaluate the indefinite integrals
(a) $\int x\sin x\,dx$ (b) $\int xe^x dx$ (c) $\int \ln x\ dx$ (d) $\int x^2\sin x dx$ (e) $\int x^2e^x dx$
[For (d) and (e), *you may need to carry out integration by parts more than once*.]

Solution (a): $\int udv = uv - \int vdu$

$u = x,$ $dv = \sin x\ dx$ $\frac{du}{dx} = 1,$ $\frac{dv}{dx} = \sin x$ [← *Sometimes this line is skipped*.] $du = dx,$ $v = -\cos x$ $\therefore \int x\sin xdx = -x\cos x + \int \cos xdx = -x\cos x + \sin x + C$	Decide on $u =$ □ $dv =$ □ Then obtain $du =$ □ $v =$ □

Solution (b): $\int xe^x dx$

$u = x,$ $dv = e^x dx$ $\frac{du}{dx} = ?,$ $\frac{dv}{dx} = ?.$ [← *Sometimes this line is skipped*.] $du = dx, v = e^x$ $\therefore \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$	Then obtain $du =$ □ $v =$ □ The choice of $u =$ □ $dv =$ □ will determine if you'll be successful in integration by parts.

Integration by parts is a technique for integrating the product of two functions; one of them can be easily integrated and the other needs to be differentiated, leading to a new form involving an integral that is easier to evaluate.

Sometimes integration by parts needs to be applied repeatedly.
Some people would introduce "**Tabular Integration**" for doing integration by parts repeatedly. Some students would end up learning only the tabular form. This is not advisable. <u>Students are advised against this</u>.

(nby, Nov 2015)